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ON THE HYPERSURFACE $x + x^2y + z^2 + t^3 = 0$ IN $\mathbf{C}^4$ OR A $\mathbf{C}^3$-LIKE THREEFOLD WHICH IS NOT $\mathbf{C}^3$

BY

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To the memory of Shimshon Amitsur

ABSTRACT

In this note it will be proved that the threefold in $\mathbb{C}^4$ which is given by $x + x^2y + z^2 + t^3 = 0$ is not isomorphic to $\mathbb{C}^3$. Here $\mathbb{C}$ is the field of complex numbers.

1. Introduction

It was first conjectured in print in 1979 (see [Kal]) that an algebraic action on $\mathbb{A}^n$ of a reductive algebraic group is linearizable, which means that with an appropriate choice of coordinate system the action is linear. It is indeed so ([Kal]) for $\mathbb{A}^2$ and it is wrong for dimensions higher than three ([Sc]). It is still not clear what is the situation for $\mathbb{A}^3$.

The simplest set-up for the linearizing conjecture is $\mathbb{C}^*$ acting on $\mathbb{C}^n$ which is of special interest (e.g. see survey [K]). This is the case which motivates this paper. Here it means that with an appropriate choice of coordinate system the action

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is given by $(x_1, x_2, \ldots, x_n) \to (\lambda^{a_1}x_1, \lambda^{a_2}x_2, \ldots, \lambda^{a_n}x_n)$, where a_i are integers. It was proved for $\mathbb{C}^2$ in [G] and it is still not known what is the situation for higher dimensions, though [Sc] contains an example of non-linearizable action of a semidirect product of $\mathbb{C}^*$ and $\mathbb{Z}/2$ on $\mathbb{C}^4$.

In a program outlined in [R1] for a clarification of the situation when $\mathbb{C}^*$ acts on $\mathbb{C}^3$, it is important to show that the contractible threefolds which are described in [R1] (some of them were found earlier in [D]; see also [K1] for a description which is actually published) are not $\mathbb{C}^3$. If any of these threefolds is $\mathbb{C}^3$ we will have an example of non-linearizable $\mathbb{C}^*$-action (see [R1] and [K2]). (This will be checked below for our particular threefold.) On the other hand, if none of them is $\mathbb{C}^3$ then some additional restrictions should take place for potential counterexamples, which raises the hope that they do not exist (see [KR]).

It was shown in [KML1], [KML2] that some of these threefolds are not isomorphic to $\mathbb{C}^3$ and even do not admit dominant mappings from $\mathbb{C}^3$. Unfortunately the technique which was developed there fails when dominant mappings exist.

From an inspection of the list of relevant examples, the threefold V which is given by $x + x^2y + z^2 + t^3 = 0$ looks like the most difficult to distinguish from $\mathbb{C}^3$. It is a smooth factorial affine variety (see [K3, Proposition 3.2]) which admits a dominant mapping from $\mathbb{C}^3$ (as was noticed by Russell). In the two-dimensional setting this implies that V is isomorphic to an affine space ([MS], [Ka2], and [R2]). It is also contractible, diffeomorphic to $\mathbb{R}^6$ ([R1], [K1]), and has negative Kodaira logarithmic dimension (see [I] for definition) as $\mathbb{C}^3$ has. It is so close to $\mathbb{C}^3$ that were it to be $\mathbb{C}^3$ it would serve as a counterexample not only to the linearizing conjecture but also to the Abhyankar–Sathaye conjecture ([A], [S]). So in a sense life would be much simpler if it would be $\mathbb{C}^3$.

Here is an explanation why an isomorphism between V and $\mathbb{C}^3$ would lead to a counterexample to the linearizing conjecture. Let us assume that V is isomorphic to $\mathbb{C}^3$ and consider the action ϕ which is given by $(x, y, z, t) \to (\lambda^6 x, \lambda^{-6}y, \lambda^3 z, \lambda^2 t)$. Let us furthermore assume that ϕ is linearizable.

So with appropriate choice of coordinates the action is given by $(u, v, w) \to (\lambda^a u, \lambda^b v, \lambda^c w)$. Since x, y, z, and t are homogeneous relative to ϕ and are not invariants of ϕ it is clear that they all belong to the augmentation ideal of $\mathbb{C}[u, v, w]$. It is also clear that the linear components of these polynomials should generate $\mathbb{C}[u, v, w]$. So at least three of these polynomials must have a non-zero linear component. If x has a linear component then $x + x^2y + z^2 + t^3 = 0$ is impossible because the linear component of x cannot be canceled. So the linear components of y, z, and t are independent and, since they are homogeneous (quasi-invariant) relative to our action, they can be used as generators u, v, w. This means that $a = -6$, $b = 3$, and $c = 2$. It is easy to check that if m is a monomial and $\phi(m) = \lambda^{-6}m$, or $\phi(m) = \lambda^3 m$, or $\phi(m) = \lambda^2 m$ then m is divisible by u, or by v, or by w correspondingly. If $\phi(m) = \lambda^6 m$ then m is divisible either by v^2 or by w^3. So we see that $y = uy_1$, $z = vz_1$, $t = wt_1$, and $x = v^2x_1 + w^3x_2$.

Let us consider now the subring I of invariants of ϕ. It is clear that $I = \mathbb{C}[uv^2, uw^3]$. On the other hand I is generated by xy, yz^2, and yt^3 with $xy + (xy)^2 + yz^2 + yt^3 = 0$. So $I = \mathbb{C}[xy, yz^2] = \mathbb{C}[xy, yt^3]$. Let us denote uv^2 by p and uw^3 by q. Then $yz^2 = y_1z_1^2p$ and $yt^3 = y_1t_1^3q$ and they are images of, say, p under some automorphisms of the polynomial ring $\mathbb{C}[p, q]$. But from the structure of the automorphism group of a polynomial ring with two generators it follows at once that if an image of p is divisible by p or q then it is c_1p, correspondingly c_2q, where $c_1, c_2 \in \mathbb{C}$ (see [C]). So $y_1, z_1, t_1 \in \mathbb{C}$ and $\mathbb{C}[y, z, t] = \mathbb{C}[u, v, w]$ which is impossible since x is not a polynomial of y, z, t.

In this note I'll prove that V is not isomorphic to $\mathbb{C}^3$. In purely algebraic terms this means that for $P = x + x^2y + z^2 + t^3$ the factor ring $R = \mathbb{C}[x, y, z, t]/(P)$ is not isomorphic to a polynomial ring in three variables. All the above-mentioned geometric motivations notwithstanding, I think that the development of algebraic tools which allow one to distinguish rings (in commutative and non-commutative settings) is an interesting and important problem in its own right.

In order to prove the claim I'll assume that R is isomorphic to $\mathbb{C}[u, v, w]$ and show that under this assumption every derivation ∂ on R satisfying the following properties:

1. ∂ is determined by a Jacobian, which means that $\partial(h) = J(f, g, h)$, where $f, g \in R$ and J denotes the determinant of the Jacobi matrix of f, g and h relative to u, v, w;

2. ∂ is locally nilpotent, which means that for each $r \in R$ there exists an $n = n(r)$ such that $\partial^n(r) = 0$;

has an additional property $\partial(X) = 0$ where X is the image of x in R. This can be easily brought to a contradiction.

2. Lemmas on locally nilpotent derivations

A derivation ∂ on a ring A is called locally nilpotent if for each $a \in A$ there exists an $n = n(a)$ such that $\partial^n(a) = 0$.

With the help of a locally nilpotent derivation acting on a ring A one can define a function $\deg_\partial$ by $\deg_\partial(f) = \min(n|\partial^n(f) = 0) - 1$ if $f \in A$ is not zero and $\deg_\partial(0) = -\infty$.

LEMMA 1: *If ∂ is a locally nilpotent nonzero derivation of a domain A then A has transcendence degree one over the subring A^∂ of constants of ∂, the field* $\mathrm{Frac}(A)$ *of fractions of A is a purely transcendental extension of* $\mathrm{Frac}(A^\partial)$, *and A^∂ is algebraically closed in A.*

Proof: Since $A \neq A^\partial$ there exists an $r \in A \setminus A^\partial$ such that $\partial(r) \in A^\partial$. It is easy to show by induction on $\deg_\partial(a) = n$ that there exist elements a_i, $b \in A^\partial$ where $i = 0, 1, \ldots, n$ and $ba_0 \neq 0$ such that $ba = \sum_{i=0}^{n} a_i r^{n-i}$. So any two elements of A are algebraically dependent over A^∂ while any element of $A \setminus A^\partial$ is transcendental over A^∂. The remaining claim that $\mathrm{Frac}(A) = \mathrm{Frac}(A^\partial)(r)$ is also obvious.

LEMMA 2: *If A is a domain then $\deg_\partial$ is a degree function, i.e.* $\deg_\partial(a+b) \leq \max(\deg_\partial(a), \deg_\partial(b))$ *and* $\deg_\partial(ab) = \deg_\partial(a) + \deg_\partial(b)$.

Proof: Follows immediately from the presentation of elements of A in the proof of Lemma 1 since it is clear that $\sum_{i=0}^{n} a_i r^{n-i}$ has degree n if $a_0 \neq 0$.

Remark: This degree function (as any degree function) induces a filtration on A. It also can be extended naturally to the field of fractions of A. Our main application will be that if a product is a ∂-constant then each factor is also a ∂-constant.

Let us denote by A_n the ring $F[x_1, x_2, \ldots, x_n]$ of polynomials. Then the Jacobian $J(f_1, f_2, \ldots, f_n)$ is a derivation in any argument.

LEMMA 3: *If $\partial(h) = J(f_1, f_2, \ldots, f_{n-1}, h)$ is a nonzero locally nilpotent derivation of A_n and $g_1, g_2, \ldots, g_{n-1} \in A_n^\partial$, then the derivation $\partial_{n-1}(h) = J(g_1, g_2, \ldots, g_{n-1}, h)$ is also locally nilpotent.*

Proof: It is clear from the definition of ∂ that all $f_i \in A_n^\partial$. We may assume that the $\{g_i\}$ are algebraically independent because otherwise ∂_{n-1} is zero. Under this assumption one of the standard proofs for the basis theorem in linear algebra can be used (the so-called *replacement* proof).

Let us show that $a\partial = a_{n-1}\partial_{n-1}$ for some $a, a_{n-1} \in A_n^\partial \setminus 0$. It is clear that the $\{f_i\}$ are algebraically independent, and since the transcendence degree of A_n is n it follows from Lemma 1 that the transcendence degree of A_n^∂ is $n-1$. So any n elements of A_n^∂ are algebraically dependent. Let P_1 be an irreducible polynomial from A_n for which $P_1(f_1, f_2, \dots, f_{n-1}, g_1) = 0$. We may assume up to renumbering the $\{f_i\}$ that P_1 depends on f_1. Then

$$\begin{aligned} 0 &= J(P_1, f_2, \dots, f_{n-1}, h) \\ &= J(f_1, f_2, \dots, f_{n-1}, h)\frac{\partial P_1}{\partial f_1} + J(g_1, f_2, \dots, f_{n-1}, h)\frac{\partial P_1}{\partial g_1}. \end{aligned}$$

So derivations ∂ and ∂_1, where $\partial_1(h) = J(g_1, f_2, \dots, f_{n-1}, h)$, are proportional with coefficients from A_n^∂ which are not zeros with our choice of P_1. Now we may assume that $\partial_i(h) = J(g_1, g_2, \dots, g_i, f_{i+1}, \dots, f_{n-1}, h)$ is a nonzero derivation and that ∂_i and ∂ are proportional over A_n^∂. Let us consider an irreducible polynomial P_{i+1} for which $P_{i+1}(g_1, g_2, \dots, g_i, f_{i+1}, \dots, f_{n-1}, g_{i+1}) = 0$. Since the elements $g_1, g_2, \dots, g_i, f_{i+1}, \dots, f_{n-1}$ are algebraically independent, such a polynomial exists and since the elements $\{g_i\}$ are algebraically independent this polynomial depends on at least one of the f's. So up to renumbering we can replace f_{i+1} by g_{i+1} and obtain ∂_{i+1}, which is not zero and proportional to ∂_i over A_n^∂. This proves the lemma.

Remark: As we can see from the proof the derivations ∂ and ∂_{n-1} have the same constants and even induce the same degree function, provided $g_1, g_2, \dots, \dots, g_{n-1}$ are algebraically independent.

Let A be a ring with $\mathbb{Z}$ filtration $\{A_i\}$ and let ∂ be a derivation on A for which $\partial(A_i) \subset A_{i+k}$ for a fixed k and all i. Let $\mathrm{Gr}(A) = \bigoplus A_i/A_{i-1}$ be the corresponding graded ring and let $h \in A_i/A_{i-1}$. Let us write $h = a + A_{i-1}$ where $a \in A_i$. We can define a homomorphism ∂_1 on $\mathrm{Gr}(A)$ which acts on h by $\partial_1(h) = \partial(a) + A_{i+k-1} \in A_{i+k}/A_{i+k-1}$ and then extend ∂_1 on $\mathrm{Gr}(A)$ by linearity. It is clear that ∂_1 is a derivation of $\mathrm{Gr}(A)$.

LEMMA 4: *If ∂ is a locally nilpotent derivation on A then ∂_1 is a locally nilpotent derivation on* $\mathrm{Gr}(A)$.

Proof: Let us denote by gr the natural mapping of A into $\mathrm{Gr}(A)$. Let $a \in A$. It is clear that either $\partial_1(\mathrm{gr}(a)) = 0$, which means that $a \in A_i$ and $\partial(a) \in A_{i+k-1}$

for some i, or $\partial_1(\mathrm{gr}(a)) = \mathrm{gr}(\partial(a))$, which means that $a \in A_i$ and $\partial(a) \in A_{i+k}$ for some i. Iterating this computation we see that either $\partial_1^n(\mathrm{gr}(a)) = 0$ or $\partial_1^n(\mathrm{gr}(a)) = \mathrm{gr}(\partial^n(a))$. Since ∂ is locally nilpotent it implies that ∂_1 is locally nilpotent on all elements from $\mathrm{gr}(A)$ and therefore on $\mathrm{Gr}(A)$.

3. Technical lemmas

LEMMA 5: *Let $y_1, y_2, \ldots, y_n$ be a set of elements in $A_{n-1} = F[x_1, x_2, \ldots, x_{n-1}]$ which generate A_{n-1} and let P be an irreducible polynomial relation between y's. Let P_i be the corresponding partial derivatives of P and J_i be the Jacobians of $y_1, y_2, \ldots, y_n$ with y_i skipped. Then $J_i = (-1)^i cP_i$ where c is a nonzero element of the ground field F.*

Proof: Since each x_i can be expressed as a polynomial of y's it is clear that $1 = J(x_1, x_2, \ldots, x_{n-1})$ belongs to the ideal generated by $J_1, J_2, \ldots, J_n$. So we see that these Jacobians are relatively prime. Therefore we may assume without loss of generality that $J_1 \neq 0$. As in the proof of Lemma 3, by computing $J(y_2, \ldots, y_{i-1}, P, y_{i+1}, \ldots, y_n) = 0$ we obtain a relation $P_i J_1 + (-1)^i P_1 J_i = 0$. So J_1 must divide P_1 and $P_i = (-1)^i d J_i$ where $-d = P_1/J_1$. It remains to show that $P_1, P_2, \ldots, P_n$ are relatively prime. (It should be obvious for a geometer.) Let us consider $A_n = F[Y_1, Y_2, \ldots, Y_n]$. Then the factor ring $A_n/(P)$, where $P = P(Y_1, Y_2, \ldots, Y_n)$, can be mapped onto A_{n-1} by $Y_i \to y_i$ and this mapping is an isomorphism since P is irreducible and the mapping is onto. Let $X_1, X_2, \ldots, X_{n-1}$ be any preimages of $x_1, x_2, \ldots, x_{n-1}$. Then we can write $Y_i = f_i + Pg_i + P^2 h_i$ where f_i and g_i are polynomials in X's. Therefore $1 = J(Y_1, Y_2, \ldots, Y_n) \equiv \sum_{i=1}^n g_i J(f_1, \ldots, f_{i-1}, P, f_{i+1}, \ldots, f_n) \pmod{P}$. But $J(f_1, \ldots, f_{i-1}, P, f_{i+1}, \ldots, f_n) = a_i J(X_1, X_2, \ldots, X_{n-1}, P)$ where $a_i \in A_n$. So

$$J(X_1, X_2, \ldots, X_{n-1}, P) \sum_{i=1}^n g_i a_i \equiv 1 \pmod{P}$$

and since $J(X_1, X_2, \ldots, X_{n-1}, P)$ is a linear combination of P_i's they are relatively prime $\operatorname{mod} P$ as elements of A_n. So their images in A_{n-1} are relatively prime. This means that d is a nonzero element of the ground field.

Remark: We may assume that $d = 1$ since we can replace the x's by another set of generators.

LEMMA 6: *Let $F_n = F(x_1, x_2, \ldots, x_n)$. Assume that we have a degree function* deg *on F_n given by $\deg(x_i) = d_i$ where $d_i \in \mathbb{Z}$ and the corresponding $\mathbb{Z}$ filtration. Let B be a subring of F_n which contains m algebraically independent elements. Then $\mathrm{Gr}(B)$ contains m algebraically independent elements.*

Proof: Let us denote by $\mathrm{gr}(a)$ the image of a in $\mathrm{Gr}(F_n)$. Let us take a maximal possible set of elements $a_1, a_2, \ldots, a_k \in B$ for which the elements $\mathrm{gr}(a_i)$ are algebraically independent. If $k = m$ the lemma is proved. Otherwise let us take an element $a_{k+1} \in B$ and elements $a_{k+2}, \ldots, a_n \in F_n$ so that $J(a_1, a_2, \ldots, a_n) \neq 0$. Let us introduce a function def on $B \setminus 0$ by $\mathrm{def}(r) = \deg(r) - \deg(J(a_1, a_2, \ldots, a_k, r, a_{k+2}, \ldots, a_n))$. This function is not identically ∞ on B and is bounded from below by $\sum_{i \neq k+1}(d_i - \deg(a_i)) + d_{k+1}$ since $\deg(J(a_1, a_2, \ldots, a_k, r, a_{k+2}, \ldots, a_n)) \leq \sum_{i \neq k+1}(\deg(a_i) - d_i) + \deg(r) - d_{k+1}$. So we can find its minimum on B, say, at element s.

The elements $\mathrm{gr}(a_1), \ldots, \mathrm{gr}(a_k), \mathrm{gr}(s)$ are algebraically dependent. So there exists a polynomial Q for which $Q(\mathrm{gr}(a_1), \ldots, \mathrm{gr}(a_k), \mathrm{gr}(s)) = 0$. We may assume that it has minimal possible degree relative to the last variable. This degree is positive since $\mathrm{gr}(a_1), \ldots, \mathrm{gr}(a_k)$ are algebraically independent.

Let $t = Q(a_1, \ldots, a_k, s)$. Clearly $t \neq 0$ (the elements $a_1, \ldots, a_k, s$ are algebraically independent since $J(a_1, a_2, \ldots, a_k, s, a_{k+2}, \ldots, a_n) \neq 0$). Now

$$\begin{aligned}\mathrm{def}(t) &= \deg(t) - \deg(\frac{\partial Q}{\partial s} J(a_1, a_2, \ldots, a_k, s, a_{k+2}, \ldots, a_n)) \\ &= \mathrm{def}(s) + \deg(t) - (\deg(\frac{\partial Q}{\partial s}) + \deg(s)).\end{aligned}$$

So if we show that $\deg(\frac{\partial Q}{\partial s}) + \deg(s) > \deg(t)$ we will get a contradiction because then $\mathrm{def}(t) < \mathrm{def}(s)$.

Let us present $Q(a_1, \ldots, a_k, s)$ as the sum of monomials: $Q(a_1, \ldots, a_k, s) = \sum_{\mathbf{i}} m_{\mathbf{i}}$ where $\mathbf{i}$ is a multi-index. As usual let $A_d = \{f \in F_n | \deg(f) \leq d\}$. Let us choose minimal d for which A_d contains $\mathrm{gr}(m_{\mathbf{i}})$ for all monomials in Q. Then the condition $Q(\mathrm{gr}(a_1), \ldots, \mathrm{gr}(a_k), \mathrm{gr}(s)) = 0$ means that $t = Q(a_1, \ldots, a_k, s)$ belongs to A_{d-1} and so $\deg(t) < d$. Our choice of Q also implies that

$$\frac{\partial Q}{\partial s}(\mathrm{gr}(a_1), \ldots, \mathrm{gr}(a_k), \mathrm{gr}(s)) \neq 0.$$

It is clear now that the monomials in $\frac{\partial Q}{\partial s}$ belong to $A_{d-\deg(s)}$ and

$$\deg(\frac{\partial Q}{\partial s}(a_1, \ldots, a_k, s)) = d - \deg(s).$$

Therefore $\deg(\frac{\partial Q}{\partial s}(a_1,\ldots,a_k,s))+\deg(s)=d>\deg(t)$. This contradiction shows that $k=m$.

Remark: It is clear from the proof that an element a_1 can be chosen arbitrarily in $B\setminus F$.

4. Application

We are going to check now that for the polynomial $P=x+x^2y+z^2+t^3$ the factor ring $R=\mathbb{C}[x,y,z,t]/(P)$ is not isomorphic to a polynomial ring in three variables. The ring R can be identified with a subring of the field $\mathbb{C}_3=\mathbb{C}(x,z,t)$ generated by x,z,t and $(x+z^2+t^3)x^{-2}$. We'll be using this identification.

Let us assume that R is $\mathbb{C}[u,v,w]$ for some $u,v,w\in\mathbb{C}_3$. Then $J_{u,v,w}(x,z,t)=x^2$ by Lemma 5 and, by the chain rule, $J_{u,v,w}(f,g,h)=x^2J_{x,z,t}(f,g,h)$. So from now on all Jacobians without subscripts will be relative to x, z, and t.

Each element $r\in R$ is a Laurent polynomial in x with coefficients in $\mathbb{C}[z,t]$. Let us introduce a degree function on $\mathbb{C}_3$ by $\deg(x)=-1$ and $\deg(z)=\deg(t)=0$. This function induces a filtration on $\mathbb{C}_3$.

Lemma 7: *If $f,g\in R$ and $\partial(h)=x^2J(f,g,h)$ is a locally nilpotent nonzero derivation on R, then* $\deg(f)\le 0$ *and* $\deg(g)\le 0$.

Proof: We may assume that $\deg(f)>0$. Then $\operatorname{gr}(f)=(\operatorname{gr}(y))^n x^m f_1(z,t)$ where $\operatorname{gr}(y)=(z^2+t^3)x^{-2}$, and $n>0$, $m\ge 0$. By Lemma 3 we could replace f and g by any two algebraically independent elements of A^∂ without changing properties of ∂. By Lemma 6 and Remark to Lemma 6 (applied to $B=R^\partial$) we may assume by replacing g if necessary that $\operatorname{gr}(g)$ and $\operatorname{gr}(f)$ are algebraically independent. So $\partial_1(h)=x^2J(\operatorname{gr}(f),\operatorname{gr}(g),h)$ is a non-trivial derivation on $\operatorname{Gr}(R)$ which is locally nilpotent on $\operatorname{Gr}(R)$ by Lemma 4 (with $k=-1$). Since $\partial_1(\operatorname{gr}(f))=0$, by Lemma 2, $\partial_1(\operatorname{gr}(y))=\partial_1(x^m)=\partial_1(f_1)=0$.

If $m>0$ then $\partial_1(x)=0$ and in this case $\partial_1(z^2+t^3)=0$. But then $\partial_2(h)=x^2J(x,z^2+t^3,h)$ is also a nilpotent derivation on $\operatorname{Gr}(R)$ (see Lemma 3 again). This would imply that $\partial_3(h)=J_{z,t}(z^2+t^3,h)$ is a locally nilpotent derivation on $\mathbb{C}[z,t]$. Since $\partial_3(z)=-3t^2$ and $\partial_3(t)=2z$, the degrees of z and t induced by this derivation should satisfy $\deg(z)-1=2\deg(t)$ and $\deg(t)-1=\deg(z)$. Hence $\deg(z)=-3$ and $\deg(t)=-2$ which is impossible.

So we may assume that $m=0$ and that $\operatorname{gr}(g)$ is also not divisible by x because otherwise $\partial_1(x)=0$ and it would imply as above that ∂_2 is locally nilpotent.

Therefore $\operatorname{gr}(g) = \operatorname{gr}(y)^k g_1(z,t)$. If $f_1, g_1 \in \mathbb{C}$ then $\operatorname{gr}(f)$ and $\operatorname{gr}(g)$ are dependent contrary to our assumption. So one of them is not in $\mathbb{C}$. Let us denote it by f_2. By Lemma 3, $\partial_2(h) = x^2 J(\operatorname{gr}(y), f_2, h)$ is a nilpotent derivation of $\operatorname{Gr}(R)$.

Let us denote $\operatorname{Gr}(R)$ by S and introduce a new degree function on S by $\deg(x) = 0$, $\deg(z) = 3$, and $\deg(t) = 2$. To distinguish the filtration which corresponds to this function from the previous one let us use the notations Gr_1 and gr_1.

If we define $\partial_3(h) = x^2 J(\operatorname{gr}(y), \operatorname{gr}_1(f_2), h)$, it is a nonzero derivation on $\operatorname{Gr}_1(S)$ which is locally nilpotent by Lemma 4. Since $\operatorname{gr}_1(f_2)$ is a homogeneous form from $\mathbb{C}[z,t]$ it has as a factor either z, or t, or $z^2 + ct^3$ where $c \in \mathbb{C}$. Since $\partial_3(\operatorname{gr}_1(f_2)) = 0$ we see from Lemma 2 that either $\partial_3(z) = 0$, or $\partial_3(t) = 0$, or $\partial_3(z^2 + ct^3) = 0$. Therefore by Lemma 3 one of the derivations $\partial_4(h) = x^2 J(\operatorname{gr}(y), z, h)$ or $\partial_5(h) = x^2 J(\operatorname{gr}(y), t, h)$ or $\partial_6(h) = x^2 J(\operatorname{gr}(y), z^2 + ct^3, h)$ is locally nilpotent.

Now

$$\begin{aligned}
&\partial_4(x) = -3t^2, &&\partial_4(z) = 0, &&\partial_4(t) = -2\operatorname{gr}(y)x;\\
&\partial_5(x) = 2z, &&\partial_5(z) = 2\operatorname{gr}(y)x, &&\partial_5(t) = 0;\\
&\partial_6(x) = 6(c-1)zt^2, &&\partial_6(z) = 6c\operatorname{gr}(y)xt^2, &&\partial_6(t) = -4\operatorname{gr}(y)xz
\end{aligned}$$

and we may assume that $c \neq 0$ and that $c \neq 1$.

Let us denote by d_x, d_z, and d_t the degrees of x, z, and t induced by these derivations. Then taking into account that the degree of $\operatorname{gr}(y)$ is zero, we'll obtain correspondingly:

$$\begin{aligned}
&d_x - 1 = 2d_t, &&d_z = 0, &&\text{and } d_t - 1 = d_x;\\
&d_x - 1 = d_z, &&d_z - 1 = d_x, &&\text{and } d_t = 0;\\
&d_x - 1 = d_z + 2d_t, &&d_z - 1 = d_x + 2d_t, &&\text{and } d_t - 1 = d_x + d_z.
\end{aligned}$$

These systems do not have nonnegative solutions, so these derivations are not locally nilpotent.

This finishes the proof of the lemma.

LEMMA 8: *If $\partial(h) = x^2 J(f,g,h)$ is a locally nilpotent nonzero derivation on R then $\partial(x) = 0$.*

Proof: Since $f, g \in \mathbb{C}[x,z,t]$ by the previous lemma, $\partial(h)$ is a locally nilpotent nonzero derivation on $\mathbb{C}[x,z,t]$. If deg is the degree function induced by this derivation and $\partial(h) \neq 0$ then $\deg(\partial(h)) \geq 2\deg(x)$. So $\deg(x)$ must be equal to zero.

To get the final contradiction let us use again Jacobians relative to u, v, and w. It is clear that each of the derivations $\partial_1(h) = J(u,v,h)$, $\partial_2(h) = J(u,w,h)$, and $\partial_3(h) = J(w,v,h)$ is nonzero and locally nilpotent and that only complex numbers are common constants of these derivations. Since $x \notin \mathbb{C}$ our assumption that $R = \mathbb{C}[u,v,w]$ is wrong.

Added in proofs: The author found out that Lemma 2 appeared as Lemma 2 in the paper of M. Ferrero, Y. Lequian and A. Nowicki, *A note on locally nilpotent derivations*, Journal of Pure and Applied Algebra **79** (1992), 45–50.

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